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Towards integrability of topological strings I: three-forms on Calabi-Yau manifolds

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Abstract: The precise relation between Kodaira-Spencer path integral and a particular wave function in seven dimensional quadratic field theory is established. The special properties of three-forms in 6d, as well as Hitchin's action functional, play an important role. The latter defines a quantum field theory similar to Polyakov's formulation of 2d gravity; the curious analogy with world-sheet action of bosonic string is also pointed out.

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1. Introduction

It has been suspected for a long time that the partition function of type B topological strings on Calabi-Yau manifold is related to a wave function for some quadratic field theory in higher dimensions. The most straightforward realization of this idea suggests the representation of the partition function as a holomorphic wave function in the seven dimensional theory with the phase space being the third cohomology $H^3(M,\mathbb{R})$ of the Calabi-Yau (CY) manifold M. In order to be compared with the generating function of correlators in topological string theory (partition function) this wave function should be defined in the linear polarization of the symplectic manifold $H^3(M,\mathbb{R})$ associated with some reference complex structure on M. The dependence on the reference holomorphic structure is governed by the Holomorphic Anomaly equation [1, 2, 3]. However, even after the 7d theory is identified, the wave function is not unique and the problem of the construction of the appropriate wave function remains.

The explicit proposal for the B-model target space field theory action was given in [2], where the generating function of correlators in the topological theory was represented as a path integral for certain field theory on the target space Calabi-Yau manifold. The critical points of this action correspond to the solutions of the Kodaira-Spencer (KS) equation describing the deformations of the complex structures on M. However the role played by this path integral in the above interpretation in terms of the holomorphic wave function was unclear.

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The obvious goal of all these studies is to find the proper, background independent, target space formulation for topological strings (target space string field theory) which hopefully will be effective enough to lead to the exact solution of the theory — integrability.

In this paper we establish the precise relation between the KS functional integral [2] and the wave function of the quadratic field theory in seven dimensions (in the companion paper [4] we will discuss the steps towards the integrability based on lessons learned here). We study two natural polarizations, linear and non-linear, in the phase space of 7d theory and construct integral transformation connecting the wave functions in these two polarizations. This result is exact, at least in the quasi-classical approximation. It appears that the path integral representation of Kodaira-Spencer quantum field theory of [2] coincides with this integral transformation for a choice of a very simple wave function in the non-linear polarization. This non-linear polarization for the symplectic space $H^3(M,\mathbb{R})$ is constructed using the special properties of the three-forms in six-dimensional space (see [5, 6, 7]) and does not depend on any reference complex structure on M. One can think about such construction as a step towards background independent formulation of the theory.[1]

It is interesting to note that the same quadratic 7d theory turns out to be useful for the description of the theory of self-dual three-forms in 6d. In this case another linear polarization (based on Hodge $*$-operation) is more adequate; for example the considerations in [8, 9] are close to those presented below.

The relation between the wave function in 7d and 6d ("chiral") path integral brings the natural question — what is the interpretation of the corresponding "non-chiral counterpart" entering the scalar product of the wave functions with the canonical measure? We briefly discuss this question at the end of the paper and propose the interpretation via the (modified) Hitchin functional — a possible six dimensional counterpart of the Polyakov formulation for the 2d gravity where chiral part (conformal block) has well-known interpretation in terms of wave-function in 3d Chern-Simons theory.

The plan of the paper is as follows. In section 1 we start with the brief description of the Kodaira-Spencer field theory formulated in [2]; we rewrite the action in terms of the variables that will be useful later. In section 2 we quantize the 7d quadratic theory and demonstrate the role played by KS path integral (written in above variables) as a wave function in the linear polarization, canonically related to a simple wave function in non-linear polarization. In the appendix A we present the main, useful, facts of the geometry of three-forms following Hitchin. In appendix B some curious analogy between Nambu-Goto/Polyakov action in two dimensions and Hitchin functional introduced in [5] is demonstrated. In appendix C we give a simple derivation of the Holomorphic Anomaly equation [1, 2], governing the dependence of the wave function in the linear polarization on the base point of the moduli space. Our derivation is based on the general properties of the special geometry of the moduli space of complex structures.

Some results presented in this paper were known to the authors for a period of time, originating to mid 90's, and were presented in the talks on various occasions [10, 11].

[1] Let us stress that the ambiguities of the quantization in the non-linear polarization might lead to the corrections to KS action beyond the quasi-classical approximation, however the assumption of the locality of the action in the natural coordinates may be used in order to fix KS action unambiguously.

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2. Geometry

2.1 Kodaira-Spencer theory

In this section we recall some facts about Kodaira-Spencer theory following [2] and rewrite the action in terms of the variables appropriate for further considerations.

Let M be a compact Calabi-Yau (CY) manifold. Calibrated CY manifold is a pair (M, Ω_0) where M is a CY manifold supplied with a holomorphic $(3,0)$-form Ω_0. Holomorphic $(3,0)$-form on M is uniquely defined up to the multiplication by a non-zero complex number and thus the moduli space $\widehat{\mathcal{M}}_M$ of the calibrated CY manifolds is a $\mathbb{C}^*$-bundle over the moduli space $\mathcal{M}_M$ of complex structures on M. Fixing particular holomorphic $(3,0)$-form Ω_0 defines, locally, a section of the bundle. In the vicinity of the point of the maximal degeneration of the complex structures the existence of the weight filtration on the homology leads to the natural choice of the three-cycle $\gamma_0 \in H_3(M, \mathbb{Z})$ and thus to the natural normalization condition $\int_{\gamma_0} \Omega_0 = 1$ [12]. Another way to normalize the holomorphic three-form is to use the condition $\int_M \Omega_0 \wedge (\overline{\Omega_0})_m = 1$ where $(\Omega_0)_m$ is a fixed holomorphic $(3,0)$-form for the reference complex structure $m \in \mathcal{M}_M$.

The choice of the complex structure provides the decomposition of the exterior derivative: $D = D^{1,0} + D^{0,1} = \partial + \bar{\partial}$. Below we will use the notation: $\Omega^{-p,q}(M) \equiv \Omega^q(M, \wedge^p T)$, where T is a holomorphic tangent bundle.

Let $(z^i, \bar{z}^{\bar{i}})$ be local coordinates on M and let $\overline{A} \in \Omega^{-1,1}(M)$ be a $(-1,1)$-differential written locally as: $\overline{A} = \sum \overline{A}^j_{\bar{i}} d\bar{z}^i \frac{\partial}{\partial z^j}$. Then the deformation of the complex structure may be described in terms of the deformation of the operator $D^{0,1} = \bar{\partial}$:

$$\bar{\partial} \to \bar{\partial}_{\overline{A}} = \bar{\partial} + \overline{A} = \sum d\overline{z}^i \left(\frac{\partial}{\partial \bar{z}^i} + \overline{A}^j_{\bar{i}} \frac{\partial}{\partial z^j} \right), \tag{2.1}$$

subjected to the integrability condition $\bar{\partial}^2_{\overline{A}} = 0$ (Kodaira-Spencer equation). This is equivalent to the following equation:

$$\bar{\partial}\overline{A} + \frac{1}{2}[\overline{A}, \overline{A}] = 0\,, \tag{2.2}$$

or explicitly

$$(\bar{\partial}_{\bar{i}} \overline{A}^k_{\bar{j}} + \overline{A}^l_{\bar{i}} \partial_l \overline{A}^k_{\bar{j}}) d\bar{z}^i \wedge d\bar{z}^j \frac{\partial}{\partial z^k} = 0\,. \tag{2.3}$$

The moduli space of complex structures is given by the space of the solutions of the equation (2.2) modulo the infinitesimal gauge transformations

$$\delta\overline{A} = \bar{\partial}_{\overline{A}} \epsilon = \bar{\partial}\epsilon + [\overline{A}, \epsilon]\,, \tag{2.4}$$

where $\epsilon \in \Omega^{-1,0}(M)$. Equivalently the moduli space of the complex structures may be parameterized by the solutions of the pair of the equations:

$$\begin{aligned} \bar{\partial}\overline{A} + \frac{1}{2}[\overline{A}, \overline{A}] &= 0\,, \\ \partial A &= 0\,, \end{aligned} \tag{2.5}$$

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modulo the subgroup of the gauge transformations (2.4) which leave the holomorphic three-form Ω_0 invariant. Note that (2.5) is equivalent to the deformation of the pair of the operators $(\bar\partial, \partial)$:

$$\begin{aligned} \bar\partial &\to \bar\partial + \overline{A}\,, \\ \partial &\to \partial\,, \end{aligned} \tag{2.6}$$

with the relations:

$$\bar\partial_{\overline{A}}^2 = \partial^2 = \bar\partial_{\overline{A}}\partial + \partial\bar\partial_{\overline{A}} = 0\,. \tag{2.7}$$

Given a holomorphic $(3,0)$-form Ω_0 we could identify $\Omega^{-p,q}(M)$ with $(3-q,p)$-forms $\Omega^{3-p,q}(M)$. The interior product $v \vdash \omega$ of the vector field $v \in Vect_M$ and the differential n-form ω is defined as: $(v \vdash \omega) = \sum_{a_1\ldots a_n}(-1)^k v_k^{a_k}\omega_{a_1\ldots a_k\ldots a_n} dx^{a_1}\wedge\ldots\widehat{dx^{a_k}}\ldots\wedge dx^{a_n}$. Extending this operation to the polyvector fields we obtain the map:

$$\begin{aligned} \Omega^{-p,q}(M) &\to \Omega^{3-p,q}(M)\,, \\ A &\to A^\vee = A \vdash \Omega_0\,. \end{aligned} \tag{2.8}$$

Under this identification the equation (2.5) reads:

$$\bar\partial\overline{A}^\vee + \frac{1}{2}\partial(\overline{A}\wedge\overline{A})^\vee = 0\,, \tag{2.9}$$

$$\partial\overline{A}^\vee = 0\,, \tag{2.10}$$

and the infinitesimal gauge transformation (2.4) is:

$$\delta\overline{A}^\vee = \bar\partial\,\epsilon^\vee + \partial(\overline{A}\wedge\epsilon)^\vee\,,$$

where $\epsilon^\vee$ is constrained by the condition $\partial\epsilon^\vee = 0$.

Let us define the following operations on the forms:

$$\begin{aligned} A^\vee \circ B^\vee &= (A\wedge B)\vdash\Omega_0\,, \\ \langle A^\vee, B^\vee, C^\vee\rangle &= A^\vee\wedge(B^\vee\circ C^\vee)\,, \end{aligned}$$

where $A^\vee = A\vdash\Omega_0$, $B^\vee = B\vdash\Omega_0$, $C^\vee = C\vdash\Omega_0$. Then the system of the equations (2.5) becomes equivalent to:

$$\begin{aligned} \bar\partial\overline{A}^\vee + \frac{1}{2}\partial(\overline{A}^\vee\circ\overline{A}^\vee) &= 0 \\ \partial\overline{A}^\vee &= 0\,. \end{aligned} \tag{2.11}$$

From the second equation in the case of the compact Kähler manifold one has:

$$\overline{A}^\vee = x + \partial b\,, \tag{2.12}$$

where x is a ∂-harmonic $(2,1)$-form and $b \in \Omega^{1,1}(M)$. Now the equation (2.11) becomes the equation for b:

$$\bar\partial\partial b + \frac{1}{2}\partial((x+\partial b)\circ(x+\partial b)) = 0\,. \tag{2.13}$$

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We note that this equation has a meaning of anti-holomorphicity for following (1,2)-current:

$$\bar{J}^{(1,2)} = \bar{\partial} b + \frac{1}{2}(x + \partial b) \circ (x + \partial b)\,; \qquad \partial \bar{J}^{(1,2)} = 0\,. \tag{2.14}$$

The action functional leading to the equations of motion in the form (2.13) is

$$S_{KS}(b|x) = \int_M \left(\frac{1}{2} \partial b \wedge \bar{\partial} b + \frac{1}{6} \langle (x + \partial b), (x + \partial b), (x + \partial b) \rangle \right). \tag{2.15}$$

This action coincides with the action (2.2) introduced in [2].

$$S'_{KS}(\overline{A}^{\vee}) = \int_M \left(\frac{1}{2} \overline{A}^{\vee} \bar{\partial}\, \partial^{-1} \overline{A}^{\vee} + \frac{1}{6} \left\langle \overline{A}^{\vee}, \overline{A}^{\vee}, \overline{A}^{\vee} \right\rangle \right). \tag{2.16}$$

In order to get precise relation one should solve the constraint (2.10) and use parameterization (2.12) to resolve the non-localities in the first term in (2.16).

2.2 KS theory in terms of 7d quadratic field theory

Now we are ready to show that the functional integral with the action (2.15)

$$Z(x) = \int Db\, e^{-S_{KS}(b|x)}\,, \tag{2.17}$$

may be considered as an integral representation of some particular wave function in the seven-dimensional field theory on $M \times \mathbb{R}$ (the pruduct of M and the real line $\mathbb{R}$) with the action functional:

$$S(C) = \int_{M \times \mathbb{R}} C dC\,. \tag{2.18}$$

Here C is a real three-form and the corresponding quantum field theory may be considered as a higher dimensional generalization of the abelian Chern-Simons theory in three dimensions. Written in $6+1$ notations in local coordinates (x^i, t) on $M \times \mathbb{R}$ one has:

$$S = \int_{M \times \mathbb{R}} dt\, d^6x \left(\Omega \frac{\partial}{\partial t} \Omega + \omega_t d\Omega \right), \tag{2.19}$$

where $C = \Omega + \omega_t dt$ with Ω — a three-form component of C along M, $\omega_t\, dt$ — a two-form along M and one-form along $\mathbb{R}$.

Consider the infinite-dimensional space of real three-forms on M supplied with the symplectic structure:

$$\omega^{\mathrm{symp}}(\delta_1 \Omega, \delta_2 \Omega) = \int_M \delta_1 \Omega \wedge \delta_2 \Omega\,. \tag{2.20}$$

The phase space for (2.18) is obtained by applying the hamiltonian reduction via imposing the first class constraint:

$$d\Omega = 0\,. \tag{2.21}$$

In order to construct the wave function explicitly one needs to chose a polarization. After the polarization is chosen — the constraints are imposed by demanding that the wave function is a solution of the system of differential equations, following from (2.21). In

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the simple cases the formal solution may be given in the integral form by applying the appropriate averaging procedure.

There are several natural polarizations on the space of three-forms in six-dimensions defined in terms of the families of the lagrangian sub-manifolds. Below two polarizations, one of which is linear and another one is non-linear will be important. It appears that KS partition function (2.17) is naturally connected with wave function in the linear polarization. On the other hand the most natural expression for the wave function is in the non-linear polarization. Thus we start with the simple expression for the wave function in the non-linear polarization and then transform it to the linear polarization using the appropriate unitary transformation. Finally we construct the wave function in the constrained theory by imposing the constraints (using the averaging procedure).

Let us start with the description of the polarizations we will use. The simplest one is a linear polarization. Given a complex structure on M one can decompose the space of complex three-forms $\Omega_{\mathbb{C}}$ over the forms of the definite Hodge type:

$$\Omega_{\mathbb{C}} = \Omega^{3,0} \oplus \Omega^{2,1} \oplus \Omega^{1,2} \oplus \Omega^{0,3} \,. \tag{2.22}$$

The real forms are singled out by the reality condition: $\Omega^{0,3} = \overline{\Omega^{0,3}}$, $\Omega^{1,2} = \overline{\Omega^{2,1}}$.

The subspace $\Omega^{3,0} \oplus \Omega^{2,1}$ defines the complex lagrangian (linear) submanifold in the space of complex three-forms and thus the complex polarization of the space of real three-forms. It will be useful to parameterize this subspace as

$$\Omega_c = \rho(\Omega_0 + \overline{A} \vdash \Omega_0) \,, \tag{2.23}$$

where Ω_0 is a fixed normalized holomorphic $(3,0)$-form in a reference complex structure, ρ is a function on M and $\overline{A}$ is a $(-1,1)$-differential.

In the specific case of the three-forms in 6d there is another polarization. It may be constructed using the decomposition of the general real three-forms Ω into the sum of two decomposable forms: $\Omega = \Omega_+ + \Omega_-$. We say that the real three-form is non-degenerate if the top-dimensional form $\Omega_+ \wedge \Omega_-$ has no zeros. The space of the non-degenerate three-forms consists of two non-intersecting parts $U_+ \cup U_-$. The decomposable forms $\Omega_\pm$ are real when $\Omega \in U_+$ and are complex conjugated to each other when $\Omega \in U_-$ (see the precise description in the appendix A).

Let us consider the forms in U_- having the following decomposition

$$\Omega = \Omega_+ + \Omega_- = E^1 \wedge E^2 \wedge E^3 + \overline{E}^1 \wedge \overline{E}^2 \wedge \overline{E}^3 \,, \tag{2.24}$$

with E^i being complex one-forms. $(E^i, \overline{E}^i)$ generically produces the frame in the complexified cotangent bundle $T^*_{\mathbb{C}}M$ and $\Omega_- \wedge \Omega_+ \neq 0$. The subspace of the decomposable forms defines the lagrangian family:

$$\omega^{\text{symp}}(\delta_1\Omega_+, \delta_2\Omega_+) = \int \delta_1(E^1 \wedge E^2 \wedge E^3) \wedge \delta_2(E^1 \wedge E^2 \wedge E^3) = 0 \,. \tag{2.25}$$

Locally the decomposable three-form Ω_+ may be parameterized as:

$$\Omega_+ = \frac{1}{6}\epsilon_{ijk}\varrho(dz^i + \mu^i_{\bar{i}} d\bar{z}^{\bar{i}})(dz^j + \mu^j_{\bar{j}} d\bar{z}^{\bar{j}})(dz^k + \mu^i_{\bar{k}} d\bar{z}^{\bar{k}})$$

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$$= \varrho(\Omega_0 + \mu \vdash \Omega_0 + \frac{1}{2}\mu^2 \vdash \Omega_0 + \frac{1}{6}\mu^3 \vdash \Omega_0)$$
$$= \varrho e^{\mu\vdash}\Omega_0\,, \tag{2.26}$$

where $\mu \in \Omega^{-1,1}(M)$ and ϱ is a function on M and we use the notations $\mu^n \vdash := (\mu \vdash)^n$.

2.3 Linear polarization

Let us proceed with the description of the wave functions in the linear polarization.[2] In this polarization the constraint equations $d\Omega = 0$:

$$\begin{aligned}\bar\partial\Omega^{3,0} + \partial\Omega^{2,1} &= 0\,,\\ \bar\partial\Omega^{2,1} + \partial\Omega^{1,2} &= 0\,,\\ \bar\partial\Omega^{1,2} + \partial\Omega^{0,3} &= 0\,,\end{aligned}$$

quantum mechanically take the form of the following equations on the wave function: $\Psi(\Omega^{3,0}, \Omega^{2,1})$

$$\begin{aligned}(\bar\partial\Omega^{3,0} + \partial\Omega^{2,1})\Psi &= 0\,,\\ (\bar\partial\Omega^{2,1} + \partial\frac{\delta}{\delta\Omega^{2,1}})\Psi &= 0\,,\\ (\bar\partial\frac{\delta}{\delta\Omega^{2,1}} + \partial\frac{\delta}{\delta\Omega^{3,0}})\Psi &= 0\,.\end{aligned} \tag{2.27}$$

The formal solution of the constraints (2.27) may be written in terms of the path integral. Given an arbitrary function $\Psi_0(\Omega^{3,0}, \Omega^{2,1})$, one can construct the formal solution representing it in the form:

$$\Psi(\Omega^{3,0}, \Omega^{2,1}) = (\Pi\Psi_0)(\Omega^{3,0}, \Omega^{2,1})\,, \tag{2.28}$$

where the projection operator Π is given by:

$$\Pi = \int D\Lambda\, D\sigma\, Db\; e^{\int_M \Lambda(\bar\partial\Omega^{3,0} + \partial\Omega^{2,1})} e^{\int_M \sigma(\bar\partial\frac{\delta}{\delta\Omega^{2,1}} + \partial\frac{\delta}{\delta\Omega^{3,0}})} e^{\int_M b(\bar\partial\Omega^{2,1} + \partial\frac{\delta}{\delta\Omega^{2,1}})}. \tag{2.29}$$

The only restriction on Ψ_0 is the convergence of the integrals in (2.29). We can use the identity: $\exp(A+B) = \exp(A - \frac{1}{2}[A,B])\exp(B)$ for A, B such that: $[A,[A,B]] = [B[A,B]] = 0$ and get:

$$\Psi(\Omega^{3,0}, \Omega^{2,1}) = \int D\Lambda\, D\rho\, Db\; e^{\int_M \Lambda(\bar\partial\Omega^{3,0} + \partial\Omega^{2,1}) + \frac{1}{2}\partial b\bar\partial b + \bar\partial b\wedge\Omega^{2,1}} \Psi_0(\Omega^{3,0} - \partial\sigma, \Omega^{2,1} - \partial b - \bar\partial\sigma)\,.$$

In order to simplify this representation let us use the following parametrization of $(\Omega^{3,0}, \Omega^{2,1})$ in terms of the non-zero constant ρ_0, $(2,0)$-form χ and $(2,1)$-form $\widetilde\Omega^{2,1}$

$$\begin{aligned}\Omega^{3,0} &= \rho_0\Omega_0 + \partial\chi\,,\\ \Omega^{2,1} &= \widetilde\Omega^{2,1} - \bar\partial\chi\,.\end{aligned} \tag{2.30}$$

[2] We abuse the notations by using $\Omega^{p,q}$ for the space of the forms of the given Hodge type and for particular (p,q)-differential form.

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In these variables:

$$\begin{aligned}\Psi(\Omega^{3,0},\Omega^{2,1}) &= \Psi(\rho_0,\chi,\widetilde{\Omega}^{2,1}) \\ &= \int D\Lambda\, D\sigma\, Db\, e^{\int_M \Lambda(\partial\widetilde{\Omega}^{2,1})} e^{-\int_M(\frac{1}{2}\partial b\bar\partial b+\bar\partial b\wedge\widetilde{\Omega}^{2,1})}\times \\ &\quad\times\Psi_0(\rho_0\Omega_0-\partial\sigma,\widetilde{\Omega}^{2,1}-\partial b-\bar\partial\sigma)\,.\end{aligned} \tag{2.31}$$

Any form $\widetilde{\Omega}^{2,1}$ has ∂-Hodge decomposition:

$$\widetilde{\Omega}^{2,1} = x+\partial\lambda+\partial^\dagger\widetilde{\lambda}, \tag{2.32}$$

with x being a harmonic form ($\partial x=\partial^\dagger x=0$). We can use this representation and replace the integral over Λ by the functional delta-function, so we finally have:

$$\begin{aligned}\Psi(\Omega^{3,0},\Omega^{2,1}) &= \Psi(\rho_0,\chi,x,\lambda,\widetilde{\lambda}) \\ &= \delta(\partial\partial^\dagger\widetilde{\lambda})e^{-\int_M(\frac{1}{2}\partial\lambda\bar\partial\lambda)}\int D\sigma\, Db\, e^{-\int_M(\frac{1}{2}\partial b\bar\partial b+\bar\partial b\wedge\partial\lambda)}\times \\ &\quad\times\Psi_0(\rho_0\Omega_0-\partial\sigma,x-\partial b-\bar\partial\sigma)\,.\end{aligned} \tag{2.33}$$

The symplectic structure is constant in terms of the variables $\Omega^{p,q}$ and the scalar product of the wave functions in given by:

$$\begin{aligned}\langle\Psi_1|\Psi_2\rangle &= \int D\Omega^{3,0}\, D\Omega^{2,1}\, D\Omega^{1,2}\, D\Omega^{0,3}\, e^{\int_M(\Omega^{3,0}\wedge\Omega^{0,3}+\Omega^{2,1}\wedge\Omega^{1,2})}\times \\ &\quad\times\overline{\Psi}_1(\Omega^{0,3},\Omega^{1,2})\Psi_2(\Omega^{3,0},\Omega^{2,1})\,.\end{aligned} \tag{2.34}$$

where integration is over the real subspace $\overline{\Omega^{0,3}}=\Omega^{3,0}$, $\overline{\Omega^{1,2}}=\Omega^{2,1}$.

Such representation of the wave function in the linear polarization may be interpreted in the same fashion as the relation between 3d Chern-Simons theory and 2d CFT — the wave function is the deformed partition function in the 6d quadratic field theory of two-forms. The examples of the interesting wave functions are given by the appropriate chiral deformations of the functional integrals in quadratic theories (i.e (B,C)-systems) in 6d (holomorphic equation (2.14) can serve as the starting point for such consideration). Below we will be interested in the particular wave function described in terms of the KS path integral.

2.4 Nonlinear polarization

Now we consider the quantization in the non-linear polarization (2.24), (2.26):

$$\begin{aligned}\Omega_+ &= \frac{1}{6}\epsilon^{ijk}\varrho(dz^i+\mu^i_{\bar i}d\bar z^{\bar i})(dz^j+\mu^j_{\bar j}d\bar z^{\bar j})(dz^k+\mu^i_{\bar k}d\bar z^{\bar k}) \\ &= \varrho(\Omega_0+\mu\vdash\Omega_0+\frac{1}{2}\mu^2\vdash\Omega_0+\frac{1}{6}\mu^3\vdash\Omega_0)\,,\end{aligned} \tag{2.35}$$

and

$$\Omega_- = \overline{\Omega}_+ = \overline{\varrho}\left(\overline{\Omega}_0+\overline{\mu}\vdash\overline{\Omega}_0+\frac{1}{2}\overline{\mu}^2\vdash\overline{\Omega}_0+\frac{1}{6}\overline{\mu}^3\vdash\overline{\Omega}_0\right). \tag{2.36}$$

We realize the quantum state as a function of (ϱ,μ).

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In order to describe the scalar product in this polarization we need to find the generating function $S_0(\varrho, \mu, \overline{\varrho}, \overline{\mu})$ of the canonical transformation from a set of variables (ϱ, μ) to a set of complex conjugate variables $(\overline{\varrho}, \overline{\mu})$. This can be done by use of the standard expression for the generating function $S(Q, q)$ for canonical transformation from (p_i, q^i) to (P_i, Q^i): $\sum_i P_i \delta Q^i - \sum_i p_i \delta q^i = \delta S(Q, q)$:

$$\int_M \Omega_-(\overline{\varrho}, \overline{\mu}) \delta\Omega_+(\varrho, \mu) - \int_M \Omega_+(\varrho, \mu) \delta\Omega_-(\overline{\varrho}, \overline{\mu}) = \delta \int_M (\Omega_-(\overline{\varrho}, \overline{\mu}) \wedge \Omega_+(\varrho, \mu)) \,, \tag{2.37}$$

and therefore the scalar product is given by

$$\langle \Psi_1 | \Psi_2 \rangle = \int \mathcal{D}(\mu, \overline{\mu}, \varrho, \overline{\varrho}) e^{\int_M \Omega_-(\overline{\varrho}, \overline{\mu}) \wedge \Omega_+(\varrho, \mu)} \overline{\Psi}_1(\overline{\varrho}, \overline{\mu}) \Psi_2(\varrho, \mu) \,. \tag{2.38}$$

The exponential factor in the integration measure may be written in terms of the Hitchin functional $\Omega_+(\Omega) \wedge \Omega_-(\Omega) = -\sqrt{\lambda(\Omega)}$ (see appendix A).

Imposing the constraints in this polarization however is note quite trivial procedure. The reason for this is that the gauge symmetry:

$$\Omega \to \Omega + d\xi \,, \tag{2.39}$$

generated by constraints does not respect the polarization. The decomposition of the gauge transformed three-form Ω on the decomposable parts:

$$\Omega = \Omega_+ + \Omega_- \to \Omega + d\phi = (\Omega_+ + \delta\Omega_+(\Omega_\pm, \phi)) + (\Omega_- + \delta\Omega_-(\Omega_\pm, \phi)) \,, \tag{2.40}$$

is highly non-linear in terms of the initial $\Omega_\pm$ (see appendix A). Thus in this polarization the gauge transformation mixes "coordinates" and "momenta" in a complicated way and the constraints are given by rather complex differential operators acting on the wave function. Therefore we will use the following strategy: start with the simple, unconstrained wave-function in the non-linear polarization $\to$ transform this wave function into the corresponding wave function in the linear polarization $\to$ impose the constraints. This gives us the constraint wave function in the linear polarization whose particular form reflects the simplicity of the initial wave function in the non-linear polarization.

Let us start with the explicit transformation for the wave functions between linear and non-linear polarizations. From (2.23) and (2.35) we establish the relations:

$$\rho\Omega_0 = \varrho\Omega_0 + \frac{1}{6}\overline{\varrho}\,\overline{\mu}^3 \vdash \overline{\Omega}_0 \,, \tag{2.41}$$

$$\rho\overline{A} \vdash \Omega_0 = \varrho\mu \vdash \Omega_0 + \frac{1}{2}\overline{\varrho}\,\overline{\mu}^2 \vdash \overline{\Omega}_0 \,, \tag{2.42}$$

$$\overline{\rho}A \vdash \overline{\Omega}_0 = \overline{\varrho}\,\overline{\mu} \vdash \overline{\Omega}_0 + \varrho\frac{1}{2}\mu^2 \vdash \Omega_0 \,, \tag{2.43}$$

$$\overline{\rho}\,\overline{\Omega}_0 = \overline{\varrho}\,\overline{\Omega}_0 + \frac{1}{6}\varrho\mu^3 \vdash \Omega_0 \,. \tag{2.44}$$

As coordinates (versus momenta) in the linear polarization (2.23) we use: $(\overline{\rho}, A)$, and in the non-linear polarization: (μ, ϱ). The reason for this particular choice is the following.

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It is clear that (2.41), (2.42), (2.43) and (2.44) may be considered as perturbations of the trivial transformation by the non-linear terms. But the kernel of the trivial transformation is given by delta-function that could not be easily described in the classical approximation as an exponent of some generating function. With the proposed choice of the polarization there is no such problem because the unperturbed transformation has smooth kernel. Thus, we can rewrite the relations in (2.41)–(2.44) in the following form:

$$\begin{aligned}
\overline{\varrho} &= \overline{\rho} - \varrho\langle\mu^3\rangle, \\
\overline{\mu} &= \frac{A - \varrho\overline{\rho}^{-1}\overline{\mu}^{\vee}}{1 - \varrho\overline{\rho}^{-1}\langle\overline{\mu}^3\rangle}, \\
\rho &= \varrho + \overline{\rho}\frac{\langle(A - \varrho\overline{\rho}^{-1}\overline{\mu}^{\vee})^3\rangle}{(1 - \varrho\overline{\rho}^{-1}\langle\overline{\mu}^3\rangle)^2}, \\
\rho\overline{A} &= \varrho\mu + \overline{\rho}\frac{(A - \varrho\overline{\rho}^{-1}\overline{\mu}^{\vee})^2}{(1 - \varrho\overline{\rho}^{-1}\langle\overline{\mu}^3\rangle)},
\end{aligned}$$

where the functions $\langle\mu^3\rangle$, $\langle\overline{\mu}^3\rangle$ and $(1,-1)$-form $\overline{\mu}^{\vee}$ are defined by:

$$\begin{aligned}
\langle\overline{\mu}^3\rangle\overline{\Omega}_0 \wedge \Omega_0 &= \frac{1}{6}\overline{\Omega}_0 \wedge (\overline{\mu}^3 \vdash \overline{\Omega}_0) \\
\langle\mu^3\rangle\Omega_0 \wedge \overline{\Omega}_0 &= \frac{1}{6}\Omega_0 \wedge (\mu^3 \vdash \Omega_0), \\
\overline{\mu}^{\vee} \vdash \overline{\Omega}_0 &= \frac{1}{2}\mu^2 \vdash \Omega_0 .
\end{aligned}$$

Now for the generating function $S(A, \overline{\rho}, \mu, \varrho)$ we have:

$$\delta S(A, \overline{\rho}, \mu, \varrho) = \Theta_{\varrho}(\mu, \overline{\mu}, \varrho, \overline{\varrho})\delta\varrho + \Theta_{\mu}(\mu, \overline{\mu}, \varrho, \overline{\varrho})\delta\mu - \Theta'_{\overline{\rho}}(A, \overline{A}, \rho, \overline{\rho})\delta\overline{\rho} - \Theta'_{A}(A, \overline{A}, \rho, \overline{\rho})\delta A , \tag{2.45}$$

where:

$$\begin{aligned}
\delta\Theta &= \delta\Theta' = \omega^{\mathrm{symp}}, \\
\Theta' &= \int_M \rho(\Omega_0 + \overline{A} \vdash \Omega_0) \wedge \delta(\overline{\rho}(\overline{\Omega_0} + A \vdash \overline{\Omega}_0)), \\
\Theta &= \int_M \Omega_-(\overline{\varrho}, \overline{\mu}) \wedge \delta\Omega_+(\varrho, \mu).
\end{aligned}$$

Solution is:

$$S(A, \overline{\rho}, \mu, \varrho) = \int_M \left(\left(\overline{\rho}\varrho + \varrho^2\langle\mu^3\rangle + \frac{\langle(A\overline{\rho} - \varrho\overline{\mu}^{\vee})^3\rangle}{(\overline{\rho} - \varrho\langle\mu^3\rangle)} \right) \Omega_0 \wedge \overline{\Omega}_0 + \overline{\rho}\varrho(\mu \vdash \Omega_0) \wedge (A \vdash \overline{\Omega}_0) \right). \tag{2.46}$$

We conclude that (at least in the quasi-classical approximation) the connection between the wave functions in the linear and the non-linear polarizations is given by:

$$\Psi(\varrho, \mu) = \int D\overline{\rho}\, DA\, e^{S(A,\overline{\rho},\mu,\varrho)}\Psi(\overline{\rho}, A), \Psi(\overline{\rho}, A) = \int D\overline{\rho}\, DA\, e^{-S(A,\overline{\rho},\mu,\varrho)}\Psi(\varrho, \mu). \tag{2.47}$$

This concludes the construction of the general wave function in the unconstrained system in the non-linear polarization. We emphasize that this representation is unambiguous only

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in the semi-classical approximation and thus the meaningful part of the integration is given by the evaluations at the critical points and determinant of quadratic fluctuations around them.

In order to get the constrained wave function we transform this function into the wave function in the linear polarization and impose the constraint. This is equivalent to the explicit calculation of the following matrix element:

$$\Psi(\Omega^{3,0},\Omega^{2,1}) = \langle \Omega^{3,0},\Omega^{2,1}|\Pi|\psi\rangle\,,$$

where $\langle\Omega^{2,1},\Omega^{3,0}|$ is the coordinate eigenvector in the linear polarization, Π is the projector imposing the constraints and $|\psi\rangle$ is some state that we will now explicitly define in the non-linear polarization.

We claim that the following choice leads to desired result:

$$\psi(\overline{\varrho},\overline{\mu}) = \delta(\overline{\mu})\exp\int_M \overline{\varrho}\,. \tag{2.48}$$

The reasoning under such selection is following: it is easy to see that in the discussed approximation the wave function in the (ϱ,μ)-polarization, corresponding to (2.48), is given by:

$$\psi(\varrho,\mu) = \delta(\varrho-1)\,. \tag{2.49}$$

This corresponds to the fixing of the holomorphic volume form (stated differently — to the choice of the closed string coupling constant). Thus we have:

$$\begin{aligned}\Psi(\Omega^{3,0},\Omega^{2,1}) &= \int D\overline{\mu}D\overline{\varrho}\; e^{-S(\Omega^{3,0},\Omega^{2,1},\overline{\mu},\overline{\varrho})}\delta(\overline{\mu})\exp\int_M\overline{\varrho} &(2.50)\\ &= \delta(\Omega^{3,0}-\Omega_0)\exp\left(-\int_M\frac{1}{6}\langle\Omega^{2,1},\Omega^{2,1},\Omega^{2,1}\rangle\right). &(2.51)\end{aligned}$$

Finally, the action of the projection operator gives:

$$\begin{aligned}\Psi(\Omega^{3,0},\Omega^{2,1}) &= \Psi(\rho_0,\chi,x,\lambda,\widetilde{\lambda})\\ &= \delta(\rho_0-1)\delta(\partial\partial^\dagger\widetilde{\lambda})e^{-\int(\frac{1}{2}\partial\lambda\bar\partial\lambda)}\times\\ &\quad\times\int Db\,e^{-\int(\frac{1}{2}\partial b\bar\partial b+\bar\partial b\wedge\partial\lambda+\frac{1}{6}\langle(x+\partial b),(x+\partial b),(x+\partial b)\rangle)}\,.\end{aligned}\tag{2.52}$$

Specializing to the subspace $\lambda=0$ we obtain, up to the x-independent factor, the following integral representation for the wave function:

$$\Psi(\Omega^{3,0},\Omega^{2,1})\to\Psi(x) = \text{const}\int Db\,e^{-\int(\frac{1}{2}\partial b\bar\partial b+\frac{1}{6}\langle(x+\partial b),(x+\partial b),(x+\partial b)\rangle)}\,. \tag{2.53}$$

This representation coincides with the KS quantum field theory path integral (2.15).

3. Conclusions and further directions

The derivation of Kodaira-Spencer path integral presented above seems rather appealing from the physical point of view. This approach is based on the use of the vacuum wave

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function in the first quantized formalism (holomorphic three-form) as a fundamental variable; the connection with the seven-dimensional formulation bears obvious similarity with 2d-gravity/3d Chern-Simons relation [13, 14].[3]

The appearance of Hitchin's action functional as the gluing factor in (2.37) is not accidental (this action functional is written in appendix A in an invariant form (A.3), where we also list some of its important properties; see also appendix B). In above-mentioned analogy, 2d gravity/3d Chern-Simons relation, the Hitchin action plays the role of Liouville theory and KS path integral — Virasoro conformal block. In this respect Hitchin action can be viewed as an interesting candidate for an alternative definition of target space quantum theory. One of the nice properties of this action is that it is almost background independent — one only needs to fix the class of three-form in $H^3(M, \mathbb{R})$.

In conclusion we would like to list some obvious further directions of study. The most obvious one is to incorporate the generalized complex structures introduced by Hitchin in [15]. The generalized complex structures are described in terms of the families of the lagrangian subspaces in the fibers of the direct sum of the tangent and cotangent bundles $TM \oplus T^*M$. From the seven-dimensional perspective it turns out to be equivalent to the replacement of the three-form C by an arbitrary odd form of mixed degree with the same quadratic action as used above. The appearance of the $\mathbb{Z}_2$-grading instead of $\mathbb{Z}$-grading strongly suggests the connection with the K-theory and the formulation in terms of the Dirac operator. Corresponding Hitchin action should be viewed as a non-chiral completion of "chiral" theory corresponding to KS action for the generalized complex structures.

Special properties of three-forms in six dimensions are not unique — three-forms and four-forms in seven and eight dimensions have very similar behavior. There is an evidence in the favor of the relation between these higher dimensional theories (of three and four-forms) and the topological strings. It has been conjectured previously [10] that topologically twisted G_2 model ($N = 1$ Superconfromal theory with 7d target [16]) shall describe both the deformation of the Complex and Kähler (both A and B model) structures for the CY manifold sitting inside G_2-manifold simultaneously, though precise realization was not known (see the last section in [16] in regard to topological twist in G_2 superconformal model on world-sheet).

The other obvious direction is the search for the KS type theories for the more general (higher-dimensional, non-compact, ...) CY manifolds and even more abstract non-geometric backgrounds. In this respect one shall stress the crucial role of the decomposability condition used in this paper which is closely tied to the criticality of 6d. One the other side, for example the Plücker-Hirota equations in KP theory are just the conditions of the decomposability of the semi-infinite forms and one would wonder if such systems come to play for non-geometric and other backgrounds?

[3] Let us remark that the formulation of the gravitational theories in terms of the gauge theory (if any) usually has the problem with the proper definition of the configuration space. The condition of the positivity of the metric removes some region in the configuration space of the gauge fields. Moreover the factorization over the mapping class group $Diff(M)^+/Diff_0$ is usually added by hand [13]. Nevertheless such representations may turn out to be useful in the search of the exact solution.

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Probably the most intriguing application is related to the possible reformulation of the critical string theory, especially in the view of the fact that KS theory is a kind of string field theory [2]. The lesson we learned is that the holomorphic volume form is the natural variable for background independent formulation (the choice of correct variables is a key ingredient in finding the exact solution of such systems, or in establishing gravity/gauge theory relations).[4] More abstractly — the holomorphic volume form in KS theory may be considered as a wave function in the first quantized formalism. This interpretation implies a straightforward generalizations (compare with the similar approach to reformulating critical string theory in [18]). However it seems that holomorphic volume form could hardly be considered as a description of truly fundamental degree of freedom. In this respect we recall other important lesson learned during the study of the non-critical strings in late 80's — for those, rather special cases, the most natural formulation was obtained in terms of the 2d fermions. The important step was the construction proposed by Kontsevich [19]. It can be reformulated in terms of the simple quadratic fermionic functional integral [11] thus leading to the explicit description of the large class of the topological B models (to be compared with the old results, see for example [20], recent progress due to [21] where unified formulation and exact solution for all effectively 2d backgrounds was given, earlier work in relation to Seiberg-Witten curve [22], etc.; we also would like to note that M. Kontsevich have made several proposals in this direction over the years [23]). One interesting way to look at this is to find a direct link between the Kontsevich type representation [19] and the KS functional integral. The implications of all these will be considered in [4]. Let us only remark here that the natural setting that emerges is very much in the spirit of geometric considerations of [24]. Briefly — for complex d-dimensional CY manifold M we start with the theory on complex $d+1$ dimensional Fano manifold N together with the inclusion $M \subset N$, such that the canonical divisor of N is proportional to the class of M in N. The dynamics of the theory on N is captured by the effective theory for complex codimension one submanifold M. The complex codimension one submanifold may be considered as a complex analog of the boundary and one expects the emergence of the phenomena familiar from AdS/CFT correspondence (perhaps this can be compared to [25]).

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A. Geometry of three-forms in six dimensions

In this appendix we provide brief description of the relevant properties of three-forms in six dimensions following Hitchin [5, 6]. Let M be a six-dimensional Calabi-Yau manifold. One

[4] According to [17] this means that the dynamical variable is a complex orientation of the six dimensional manifold.

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of the special properties of three-forms in six dimensions is the possibility to decompose any generic real three-form as a sum of two (possibly complex conjugate) decomposable three-forms:

$$\Omega = \Omega_+ + \Omega_- = E^1 \wedge E^2 \wedge E^3 + E^4 \wedge E^5 \wedge E^6 \,, \tag{A.1}$$

where E^i's are some one-forms. Generically $\Omega_- \wedge \Omega_+ \neq 0$ and E^i's produce the frame in the complexified cotangent bundle $T^*_{\mathbb{C}}M$ to M. The representation (A.1) follows from the existence of the open orbit of the group $GL(V)$ acting on $\wedge^3 V^*$ where V a six-dimensional vector space.

One could explicitly reconstruct the decomposable forms $\Omega_\pm$ as follows. Let $\mathcal{A}^p_M$ be the space of the real p-forms on M and $Vect_M$ be the space of vector fields. Given a three-form Ω, consider the operator K_Ω:

$$K_\Omega : \mathrm{Vect}_M \to \mathcal{A}^5_M \simeq \mathrm{Vect}_M \otimes \mathcal{A}^6_M \,,$$

defined as:

$$v \to (v \vdash \Omega) \wedge \Omega \,.$$

For instance, given the decomposition (A.1) with real E^i's, the action of K_Ω on the dual real frame E^*_i is:

$$\begin{aligned} K_\Omega : \;\; E^*_i \;&\to E^*_i \,, \qquad i = 1,2,3 \,, \\ K_\Omega : \;\; E^*_i \;&\to -E^*_i \,, \qquad i = 4,5,6 \,. \end{aligned}$$

Now the decomposable components $\Omega_\pm$ may be defined as follows. Let K^*_Ω be the group action of K_Ω on $\mathcal{A}^p(M)$. Then:

$$\begin{aligned} 2\Omega_+ &= \Omega + \lambda(\Omega)^{-3/2} K^*_\Omega \Omega \,, \\ 2\Omega_- &= \Omega - \lambda(\Omega)^{-3/2} K^*_\Omega \Omega \,, \end{aligned}$$

where:

$$\lambda(\Omega) = \frac{1}{6} \,\mathrm{tr}\, K^2_\Omega \in (\mathcal{A}^6_M)^{\otimes 2} \,. \tag{A.2}$$

Decomposition (A.1) is non-degenerate (i.e $\Omega_+ \wedge \Omega_-$ has no zeros) if the form $\lambda(\Omega)$ has no zeros. The sign of $\lambda(\Omega)$ defines whether $\Omega_\pm$ are real ($\lambda(\Omega) > 0$) or complex conjugate to each other $\Omega_- = \overline{\Omega}_+$ ($\lambda(\Omega) < 0$). Denote the corresponding subspaces in the space of real three-forms by U_+ and U_-. In the case $\Omega \in U_-$ the operator I

$$I_\Omega = (-\lambda(\Omega))^{-1/2} K_\Omega \,,$$

defines the (pseudo)complex structure. The condition of the integrability of this complex structure may be described as

$$d\Omega = d\widehat{\Omega} = 0 \,,$$

where $\widehat{\Omega} \equiv \Omega_+ - \Omega_-$. Note that $\Omega + i\widehat{\Omega}$ is a holomorphic $(3,0)$-form without zeros in this complex structure. It turns out that the integrability condition may be formulated as an equation for the critical points of the functional $\Phi(\Omega)$ written in terms of the closed three-form Ω, $d\Omega = 0$

$$\Phi(\Omega) = \int_M \sqrt{|\lambda(\Omega)|} \,. \tag{A.3}$$

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The variation of this functional is given by:

$$\delta\Phi(\Omega) = -\int_M \widehat{\Omega} \wedge \delta\Omega \,.$$

The following relation holds

$$\Omega_+ \wedge \Omega_- = \frac{1}{2}\Omega \wedge \widehat{\Omega} = (\lambda(\Omega))^{1/2} \,.$$

Let us restrict the space of three-forms Ω by the condition: $d\Omega = 0$, and fix the class of Ω in $H^3(M, \mathbb{R})$. Such three-forms may be parameterized by two-form ϕ as: $\Omega = x + d\phi$, where x is some fixed closed three-form $[\Omega - x] = 0$ in $H^3(M, \mathbb{R})$. The critical points of the functional (A.3) under the variation $\delta\Omega = d\delta\varphi$ are given by the solutions of the equation $d\widehat{\Omega} = 0$. Given the cohomology class of the real closed three-form on M, and using the critical point condition one can reconstruct unambiguously the holomorphic structure and the holomorphic non-degenerate $(3,0)$-form $\Omega + i\widehat{\Omega}$ on M; thus M has a trivial canonical class. One can show [5] that up to the action of the diffeomorphisms the critical point is isolated, so it defines the map of the subspace of $H^3(M, \mathbb{R})$ (such that the corresponding critical value Ω is in U_-) into the moduli space $\widehat{\mathcal{M}}_M$ of calibrated CY manifold.

B. On six dimensional field theory

In this appendix we consider a field theory of two-forms that may be formulated due to the special properties of six dimensional geometry. The exact quantum construction, if exists, obviously deserves the introduction of the additional degrees of freedom. Thus our discussion will be rather formal.

Consider the following formal path integral:

$$Z = \int_{(x+d\phi)\in U_-} d\phi \int Dk\, e^{\int_M \frac{1}{\sqrt{\frac{1}{6}\,\mathrm{tr}\,k^2}}(x+d\phi)k(x+d\phi)} \,, \tag{B.1}$$

where x is some fixed element of $H^3(M, \mathbb{R})$, ϕ is a two-form and $k \in End(T^*M)$ acts on arbitrary differential form as an element of the Lie algebra. The equations of motion for k are algebraic and its solution is $k = \rho K_\Omega$ for $\Omega = (x + d\phi)$ with ρ being an arbitrary non-zero function. Substituting this solution into the action in (B.1) one finds that (in the classical approximation over k) the theory described by (B.1) is equivalent to:

$$Z = \int_{U_-} D\phi\, e^{\int_M \sqrt{-\lambda(x+d\phi)}} \,, \tag{B.2}$$

Note that (B.2) does not depend on ρ.

This should be compared with well-known procedure in two dimensions. Mainly — start with Polyakov formulation of the string moving in d dimension:

$$Z = \int \left(\prod_{a=1}^{p} d\phi^a\right) Dg_{ij}\, e^{\int_M \sqrt{g} g^{ij} \sum_{a=1}^{p} \partial_i\phi^a \partial_j\phi^a} \,, \tag{B.3}$$

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In two dimensions the analog of k can be explicitly described in terms of the metric as:

$$k_i^j = |g|\epsilon_{ik}g^{kj}\,.$$

One has $\mathrm{tr}\,k^2 = 2|g|$ (compare with (A.2)). Then the action in 2d is given by

$$S = \int_M \frac{1}{\sqrt{\frac{1}{2}\,\mathrm{tr}\,k^2}} \sum_{a=1}^{p} d\phi^a \wedge (kd\phi^a)\,.$$

One can get rid of k using its equations of motion, so result is a Nambu-Goto action:

$$S = \int_M \sqrt{\det_{i,j=1,2}\left(\sum_{a=1}^{p} \partial_i\phi^a\partial_j\phi^a\right)}\,.$$

Thus we have a remarkable analogy between the case of the two-forms in six dimensions and scalars in two dimension.

Note that the proper generalization of the metric in two dimensions in this context is given by the non-normalized operator of the complex structure k.

Let us finally remark that higher dimensional version of the case of the several scalar fields ϕ^a $a = 1, \ldots, p$, in two dimensions is presumably connected with the generalization of above approach to the d-dimensional manifolds with $N = \dim H^{d/2,0}(M) > 1$; the volume vol_M form may be represented as

$$\mathrm{vol}_M = \sum_{a=1}^{N} \Omega_a \wedge \overline{\Omega}_a$$

with Ω_a being the basis of $H^{d/2,0}(M)$.

We stress that in $6d$ case the natural variable is not the metric but non-normalized pseudocomplex structure. Both in $d = 2$ and $d = 6$ case the actual variable is the operator k acting on $d/2$ dimensional forms. Locally in $d = 2$ this operator is characterized by the normalization function. Due to the special properties of the metric in two dimensions the choice of the metric is also locally reduced to the choice of the conformal factor and thus is equivalent to the choice of the volume form. On the contrary in six dimensions the parametrization of the metric is more involved and the proper generalization is in terms of the operator k.

C. Derivation of the Holomorphic Anomaly equation

In this appendix we give a derivation of the Holomorphic Anomaly (HA) equation of [2]. Its interpretation as a heat-kernel type connection describing the change of the polarization in the quantization of $H^3(M, \mathbb{R})$ was proposed in [3] (see also [26] for some additional clarifications).

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The main ingredient in [2] is the fact that the moduli space of the (calibrated) complex CY manifolds has the structure of the (projective) special Kähler geometry introduced first in the physical context by [27] (see [28] for more rigorous presentation). Below we derive HA equations in the case of an abstract (projective) special manifold and then show how the specification to the particular case of the moduli space at hand leads to the results from [2]. In abstract terms — starting with the canonical flat connection on the tangent bundle to the special Kähler manifold we construct the flat connection on the associated flat infinite dimensional non-linear bundle of the Weyl algebras (quantization of the formal completion of the zero section of the tangent bundle). The wave functions are given by the flat sections of this connection (Holomorphic Anomaly equation). The constructed connection has interesting properties. For instance its classical approximation defines new holomorphic structure on the formal completion of zero section of the tangent bundle. This new complex structure (as was shown by Kapranov in [29]) coincides with the complex structure on the formal completion of the natural inclusion $M \to M \times \overline{M}$.

We start with the description of the local geometry of the (projective) special Kähler manifolds. The special Kähler manifold is a Kähler manifold with a flat torsion-free symplectic connection on the tangent bundle with some compatibility condition between complex and flat structures. More explicitly let I, g and ω be a complex structure, a Kähler metric and corresponding Kähler form on the manifold M. Then M admits a special Kähler structure if there is a flat torsion-free symplectic connection $\mathcal{D}$ such that the following compatibility condition holds:

$$\mathcal{D}_\mu I^\lambda_\nu = \mathcal{D}_\nu I^\lambda_\mu \,. \tag{C.1}$$

The flat connection $\mathcal{D}$ defines the covering of M by the flat Darboux coordinate systems (x^i, y_i):

$$\omega = \sum_i dx^i \wedge dy_i \,,$$

subjected to the conditions:

$$\mathcal{D}\, dx^i = \mathcal{D}\, dy_j = 0 \,.$$

Let ∂ be a holomorphic part of the exterior derivative d. From the compatibility condition (C.1) it follows that in the decomposition:

$$\partial = \sum dz^i \frac{\partial}{\partial x^i} - \sum dw_i \frac{\partial}{\partial y_i} \,,$$

the one-forms dz^i and dw_i are holomorphic and locally exact. The functions z^i may be considered as local coordinates and one has the following expression for the corresponding vector fields:

$$\frac{\partial}{\partial z^i} = \frac{1}{2}\left(\frac{\partial}{\partial x^i} - \tau_{ij}(z)\frac{\partial}{\partial y_i}\right),$$

where:

$$\tau_{ij} = \frac{\partial w_i(z)}{\partial z^j} \,. \tag{C.2}$$

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The Kähler form ω has the type $(1,1)$ and therefore τ_{ij} is symmetric:

$$0 = \omega\left(\frac{\partial}{\partial z^i}, \frac{\partial}{\partial z^j}\right) = \tau_{ij} - \tau_{ji}\,.$$

Together with (C.2) it means that locally there exists a holomorphic function $\mathcal{F}(z)$ with the property:

$$\tau_{ij}(z) = \frac{\partial^2 \mathcal{F}}{\partial z^i \partial z^j}\,.$$

The expression for the Kähler form is easily obtained by evaluating $\omega(\frac{\partial}{\partial z^i}, \frac{\partial}{\partial \overline{z}^{\bar{i}}})$. In terms of holomorphic function $\mathcal{F}$ Kähler potential K, Kähler form ω and Kähler metric g are given by:

$$\begin{aligned} K &= \sum_{i,\bar{i}} \left(z^i \frac{\partial \overline{\mathcal{F}}}{\partial \overline{z}^{\bar{i}}} M_i^{\bar{i}} - \overline{z}^{\bar{i}} \frac{\partial \mathcal{F}}{\partial z^i} M_{\bar{i}}^i \right), \\ \omega = \partial\bar{\partial} K &= \frac{\sqrt{-1}}{2} \sum_{i,\bar{j}} \mathrm{Im}\, \frac{\partial^2 \mathcal{F}}{\partial z^i \partial z^j} dz^i \wedge dz^{\bar{j}} M_{\bar{j}}^j\,, \\ g &= \frac{1}{2} \sum_{i,\bar{j}} \mathrm{Im}\, \frac{\partial^2 \mathcal{F}_0}{\partial z^i \partial z^j} dz^i \otimes dz^{\bar{j}} M_{\bar{j}}^j\,. \end{aligned} \tag{C.3}$$

Here we use the spurious matrix $M_{\bar{j}}^i = \delta_{\bar{j}}^i$, $M_i^{\bar{j}} = \delta_i^{\bar{j}}$ to make the contraction of the indexes more natural.

In the case of the special geometry the complex structure is not necessary covariantely constant with respect to the flat connection (otherwise all special Kähler manifolds would be flat). This property may be characterized by introducing the following holomorphic symmetric three-tensor

$$C_{ijk} \equiv C\left(\frac{\partial}{\partial z^i}, \frac{\partial}{\partial z^j}, \frac{\partial}{\partial z^k}\right) = -4\omega\left(\frac{\partial}{\partial z^i}, \mathcal{D}_{\frac{\partial}{\partial z^j}}\left(\frac{\partial}{\partial z^k}\right)\right) = \frac{\partial^3 \mathcal{F}}{\partial z^i \partial z^j \partial z^k}\,. \tag{C.4}$$

Then the curvature $R = D^2$ of the Levi-Civita connection $D = d + \Gamma$ for the metric g may be written as:

$$\Gamma_{ik}^j = -g^{j\bar{j}} \partial_i g_{k\bar{j}} = C_{ikl} g^{\bar{l}j} M_{\bar{l}}^l \tag{C.5}$$

$$R_{i\bar{j}l}^k = -\partial_{\bar{j}} \Gamma_{il}^k = C_{ilp} g^{p\bar{q}} \bar{C}_{\bar{q}\bar{j}\bar{m}} g^{k\bar{m}}\,. \tag{C.6}$$

The simplest case when the structure of the special Kähler geometry naturally arises is the lagrangian sub-manifolds of the flat holomorphic symplectic manifolds [7]. Let $V = T^*\mathbb{C}^n = \mathbb{C}^{2n}$ be the complex symplectic vector space supplied with the holomorphic symplectic structure and (indefinite) metric G:

$$\Omega = \sum_{i=1}^n dz^i \wedge dw_i, \tag{C.7}$$

$$G = \sqrt{-1} \sum_{i,\bar{i}=1}^n \left(dz^i \otimes d\overline{w}_{\bar{i}} M_i^{\bar{i}} - dw_i \otimes d\overline{z}^{\bar{i}} M_{\bar{i}}^i \right). \tag{C.8}$$

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The holomorphic embedding $\phi : M \to V$ such that the image of M is a lagrangian submanifold provides M with the induced structure of the pseudo-Kähler manifold. Conversely, any simply connected special Kähler manifold $(M, J, g, \mathcal{D})$ admits a Kähler lagrangian embedding $\phi : M \to V$ inducing the data $(g, \mathcal{D})$. Any holomorphic lagrangian submanifold may be locally described as a graph of the locally exact holomorphic one-form $d\mathcal{F}$. In the discussed case the holomorphic function $\mathcal{F}$ coincides with the function entering the local description of the abstract special Kähler manifold above.

In order to derive the HA equation form [2] we need to consider the special Kähler manifold supplied with the action of $\mathbb{C}^*$. This leads to the notion of the projective special manifold (see [28]) and moduli space of the calibrated CY manifolds posses exactly these properties. Thus, let M be a special manifold with free $\mathbb{C}^*$-action leaving invariant the flat connection $\mathcal{D}$ and the symplectic structure. We denote by N the corresponding quotient manifold. Also let the special holomorphic coordinates on M be of the degree one and the function $\mathcal{F}$ — of degree two with respect to the action of $\mathbb{C}^*$. Then given the special pseudo-Kähler metric on M of the signature $(n, 1)$, the induced metric on the quotient space provides N with the structure of the projective special Kähler manifold.

Let $(z^0, z^1, \dots z^n)$ be a coordinate system on M; the $\mathbb{C}^*$-action is given by: $(z^0, z^1, \dots, z^n) \to (\lambda z^0, \lambda z^1, \dots, \lambda z^n)$. The non-homogeneous coordinates: $(y^0 = z^0, y^i = z^i/z^0)$ provide the natural coordinates $(y^1, \dots, y^n)$ on N and the Kähler potential has the form:

$$K(z^0, z^1, \dots, z^n) = \mid y^0 \mid^2 k(y^1, \dots, y^n)\,,$$

with some function $k(y)$. The components of the metric are given by

$$\begin{aligned} h_{i\bar{j}} &= \partial_i \bar{\partial}_j k(y) \mid y^0 \mid^2, \qquad & h_{0\bar{0}} &= k(y)\,, \\ h_{i\bar{0}} &= \partial_i k(y) \bar{y}^0\,, & h_{0\bar{i}} &= \partial_{\bar{i}} k(y) y^0\,. \end{aligned}$$

Define new frame in the tangent space TM as:

$$v_0 = y^0 \frac{\partial}{\partial y^0}\,, \qquad v_i = \frac{\partial}{\partial y^i} + \partial_i \log k(y) y^0 \frac{\partial}{\partial y^0}\,. \tag{C.9}$$

Then the Kähler metric in this frame $g = \partial\bar{\partial} K$ on M:

$$\begin{aligned} g_{0\bar{0}} &= k(y)|y^0|^2\,, \qquad & g_{i\bar{0}} &= g_{0\bar{i}} = 0\,, \\ g_{i\bar{j}} &= \mid y^0 \mid^2 \left(\partial_i \bar{\partial}_j k(y) - \frac{\partial_i k \bar{\partial}_{\bar{j}} k}{k} \right), \end{aligned}$$

defines the special Kähler metric on the quotient space N:

$$G_{ij} = \frac{g_{ij}}{g_{00}} = \partial\bar{\partial} \log k(y^i)\,,$$

and the metric on $\mathbb{C}^*$-bundle $\mathcal{L}$:

$$\|s(y)\|^2_{\mathcal{L}} = |s(y)|^2 k(y)\,.$$

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The expression for the curvature of the metric G may be easily obtained:

$$R_{i\bar{j}l\bar{m}} = G_{i\bar{j}}G_{l\bar{m}} + G_{i\bar{m}}G_{l\bar{j}} - C_{ilp}\overline{C}_{\bar{j}\bar{m}\bar{p}}G^{p\bar{p}}\,.$$

Now let us remind the notion of the special coordinates on Kähler manifold [2] (for more rigorous treatment see [29]). Fix a point $(z^i_*, \bar{z}^{\bar{i}}_*)$ in the Kähler manifold M and let $\omega = \partial\bar{\partial}K(z,\bar{z})$ be the Kähler form. Some components of the curvature of the Levi-Civita connection in the Kähler geometry are identically zero:

$$[D_i, D_j] = 0\,.$$

This allows us for any point $(z_*, \bar{z}_*) \in M$ to chose such local coordinates $(\eta, \bar{\eta})$, based at this point $(z^i = z^i_* + \eta^i + \mathcal{O}(\eta^2))$, that all terms of the Taylor expansion of the Christoffel symbols over η^i are zero:

$$\frac{\partial^n}{\partial\eta^{i_1}\cdots\partial\eta^{i_n}}\Gamma^k_{ij}(z,\bar{z})|_{z=z_*} = 0\,.$$

Explicitly new coordinates η are given by:

$$\eta^j(z, z_*) = G^{j\bar{j}}(z_*\bar{z}_*)(\bar{\partial}_{\bar{j}}K(z,\bar{z}_*) - \bar{\partial}_{\bar{j}}K(z_*,\bar{z}_*))\,,$$

where we imply the analytic continuation of the Kähler potential. Thus for the special Kähler metric (C.3) we have

$$\mathrm{Im}(\tau_{ij}(z_*))\eta^j = (w_i(z) - w_i(z_*)) - (\overline{\tau}_*)_{ij}(z^j - z^j_*)\,, \tag{C.10}$$

where $(\tau_*)_{ij} = \tau_{ij}(z_*)$. It is useful to express the relation between the coordinates z^i and η^i in the form of the following differential equations:

$$\frac{\partial z^i(\eta; z_*)}{\partial\eta^j} = \frac{\partial z^i(\eta; z_*)}{\partial z^j_*} + \Gamma^k_{jl}(z_*, \bar{z}_*)\eta^l\frac{\partial z^i(\eta; z_*)}{\partial\eta^k}\,.$$

Each special Kähler manifold M may be considered locally as a lagrangian submanifold in the flat holomorphic symplectic manifold W. In particular W may be identified with the complexified tangent space at some fixed point in M. Let us chose some holomorphic Darboux coordinates (z^i, w_i) on W. Now given the special coordinates η^i on M based at the point $(z_*, \bar{z}_*)$ it is natural to introduce the additional coordinates ξ_i such that $(\eta^i; \xi_i)$ provides another set of Darboux coordinates in W. Namely consider the following canonical transformation from the variables (z^i, w_i) to the new variables (η^i, ξ_i)

$$\eta^i = g(z_*)^{i\bar{j}}M^j_{\bar{j}}(w_j - w_j(z_*) - \overline{\tau}_{\bar{i}\bar{j}}(z_*)M^{\bar{j}}_j(z^j - z^j_*))\,, \tag{C.11}$$

$$\xi_i = (w_i - w_i(z_*)) - \tau_{ij}(z_*)(z^j - z^j_*)\,. \tag{C.12}$$

The coordinates (η, ξ) are canonical variables and the lagrangian plain $(\xi_i = 0)$ is tangent to lagrangian subspace $\mathcal{M}$ at the point $(z_*; \overline{z}_*)$. Inverse transformation is:

$$\begin{aligned} w_i &= w(z_*)_i + M^{\bar{i}}_i\overline{\tau}_{\bar{i}\bar{j}}(z_*)g(z_*)^{k\bar{j}}\xi_k + \tau_{ik}(z_*)\eta^k\,,\\ z^i &= z^i_* + g(z_*)^{i\bar{k}}M^k_{\bar{k}}\xi_k + \eta^i\,. \end{aligned}$$

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Note that the restriction on M ($w_i = w_i(z) \equiv \frac{\partial \mathcal{F}}{\partial z^i}$) turns η^i into the special coordinates (C.10). The generating function $S(z,\eta)$ of the canonical transformation (C.11)

$$dS(z;\eta) = w_i dz^i - \xi_i d\eta^i\,,$$

is given by:

$$S(z;\eta) = \frac{1}{2}\overline{\tau}_{\bar{i}\bar{j}}(z_*)M_i^{\bar{i}}M_j^{\bar{j}}(z^i - z_*^i)(z^j - z_*^j) - \frac{1}{2}\,\mathrm{Im}(\tau_{ij}(z_*))\eta^i\eta^j + \tag{C.13}$$

$$+\,\mathrm{Im}(\tau_{ij}(z_*))\eta^i(z^j - z_*^j) + +w_i(z_*)(z^i - z_*^i) - \mathcal{F}(z_*).$$

The change of the coordinates (η,ξ) under the infinitesimal change of the base point $(z_*,\overline{z}_*)$ is given by the following canonical transformation:

$$\delta\eta^i = \Gamma^i_{kl}\eta^l\delta z_*^k - \delta z_*^i + g(z_*)^{i\bar{j}}\bar{C}_{\bar{j}\bar{l}\bar{k}}g(z_*)^{p\bar{l}}\xi_p\delta\overline{z}_*^{\bar{k}}, \tag{C.14}$$

$$\delta\xi_i = -\Gamma^l_{ki}\xi_l\delta z_*^k + C_{ijk}\eta^j\delta z_*^k. \tag{C.15}$$

Now consider the quantization of the holomorphic symplectic manifold in the polarization defined by the condition that the wave function $\Psi(\eta)$ depends only on the coordinates η^i. According to the standard quantization rules one has

$$\widehat{\eta^j} \to \eta^j,$$

$$\widehat{\xi}_j \to i\frac{\partial}{\partial\eta^j}.$$

The variation of the wave function under the change of the base point $(z_*,\bar{z}_*)$ is given by the Bogolubov transformation corresponding to (C.14). Thus we have the following compatible system of equations on the wave function $\Psi(\eta, z_*, \overline{z}_*)$:

$$\frac{\partial\Psi}{\partial z_*^i} = (\Gamma^j_{ik}\eta^k\frac{\partial}{\partial\eta^j} - \frac{\partial}{\partial\eta^i} + \frac{1}{2}C_{ijk}\eta^j\eta^k + +\frac{\partial}{\partial z^i}\log(\det\mathrm{Im}\,\tau_*))\Psi, \tag{C.16}$$

$$\frac{\partial\Psi}{\partial\overline{z}_*^i} = \bar{C}_{\bar{i}\bar{j}\bar{k}}g(z_*)^{p\bar{j}}g(z_*)^{q\bar{k}}\frac{\partial^2\Psi}{\partial\eta^p\partial\eta^q}. \tag{C.17}$$

Before we start discussing the connection of (C.16), (C.17) with Holomorphic Anomaly equation in [2] let us make some remarks.

The quasiclassical limit of the equation (C.17) defines the new complex structure on the infinite-dimensional bundle over the base M. It appears that this complex structure is induced from the natural complex structure on $\mathcal{O}(M\times\overline{M})$ in the vicinity of the diagonal $M\to M\times\overline{M}$.

Let us note that the general solution of (C.16), (C.17) may be constructed stating with an arbitrary holomorphic function $\psi(z)$ on M. It can be checked by the direct calculation that the wave function given by the following integral transformation:

$$\Psi(\eta, z_*, \overline{z}_*) = \int d^n z\left[\det\left(\frac{\partial^2 S}{\partial z^j\partial\eta^k}\right)\right]^{\frac{1}{2}} e^{\frac{i}{\hbar}S(z,\eta)}\psi(z)\,, \tag{C.18}$$

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satisfies the equation (C.16), (C.17) for an arbitrary holomorphic function $\psi(z)$. This expression is obviously correct quasi-classically and due to the linearity of the canonical transformation (C.14), (C.15) gives the exact unitary transformation of quantum wave function (the determinant term in the measure of integration correctly takes into account that wave functions are naturally half-densities on lagrangian sub-manifolds).

Suppose the holomorphic function $\psi(z)$ has the expansion:

$$\psi(z) = \exp \frac{1}{\hbar}\mathcal{F}_0(z) + \sum_k \hbar^k \mathcal{F}_k(z)\,,$$

with some holomorphic functions $\mathcal{F}_k(z)$. It easy to show that if $\mathcal{F}_0$ coincides with the function $\mathcal{F}$ entering the description of the special geometry — the wave function: $\Psi(\eta, z_*, \bar{z}_*)$ has the following expansion:

$$\mathcal{F}_0(\eta, z_*, \overline{z}_*) = \frac{1}{6}\frac{\partial^3 \mathcal{F}(z_*)}{\partial z^i \partial z^j \partial z^k}\eta^i\eta^j\eta^k + \cdots\,. \tag{C.19}$$

This should be compared to the properties of the classical KS action functional [2] (see also [30]).

In order to deduce the Holomorphic Anomaly equations of [2] we should specialize (C.16), (C.17) to the case of the projective special Kähler manifold. Note that η^i may be considered as linear coordinates on the fiber of the tangent bundle TM at $(z_*, \bar{z}_*)$ and thus define the basis E^i in the tangent space TM at the point $(z_*, \bar{z}_*)$. By using the nonhomogeneous coordinates $(y^0, \ldots, y^n)$ one can construct another bases e^i:

$$E^i = y^0 e^i + y^i e^0\,, \qquad E^0 = e^0\,,$$

such that the dual basis is given by:

$$E_i = \frac{1}{y^0}e_i\,, \qquad E_0 = e_0 - \sum_k \frac{y^k}{y^0}e_k\,.$$

Finally we define the third basis $(f^0, \ldots, f^n)$:

$$\begin{aligned} e^i &= f^i\,, \qquad & e^0 &= f^0 + \textstyle\sum_i f^i y^0 \partial_i \log k(y)\,, \\ f_0 &= e_0\,, \qquad & f_i &= e_i + y^0 \partial_i \log k(y) e_0\,. \end{aligned}$$

We would like to rewrite the equations (C.16), (C.17) in the coordinates $(y^0, \ldots, y^n)$ on the base and in the linear coordinates η_1^i in the fiber, associated with the frame $(f^0, \ldots, f^n)$. For the symmetric three-tensor C and Christoffel symbols simple calculation gives:

$$\begin{aligned} C(y)_{ijk} &= \frac{1}{(y^0)^3}C(z)_{ijk}\,, \\ C(y)_{0jk} &= C(y)_{00k} = C(y)_{000} = 0, \\ \Gamma^k_{ij}(g) &= \Gamma^k_{ij}(G) + \partial_i \log k(y)\delta^k_j\,, \qquad & \Gamma^0_{i0}(g) &= \partial_i \log k(y)\,, \\ \Gamma^0_{ij}(g) &= \Gamma^k_{i0}(g) = 0\,. \end{aligned}$$

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Finally we have the following system of equations on the wave function $\Psi(\eta_1, y_*, \overline{y}_*)$:

$$\begin{aligned}\frac{\partial\Psi}{\partial y_*^i} &= \left(\Gamma_{il}^j\eta^l\frac{\partial}{\partial\eta_1^j} - \frac{\partial}{\partial\eta_1^i} + \partial_i\log k(y)\left(y_*^0\frac{\partial}{\partial{\eta_1}^0} - \eta_1^0\frac{\partial}{\partial\eta_1^0} - \eta_1^i\frac{\partial}{\partial\eta_1^i}\right) + \right.\\ &\left. + \frac{1}{2}(y_*^0)^2C(y)_{ijk}\eta_1^j\eta_1^k + \frac{\partial}{\partial y_*^i}\log(\det\operatorname{Im}\tau_*)\right)\Psi\,,\\ y_*^0\frac{\partial\Psi}{\partial y_*^0} &= \left(y_*^0\frac{\partial}{\partial\eta_1^0} - \eta_1^0\frac{\partial}{\partial\eta_1^0} - \eta_1^i\frac{\partial}{\partial\eta_1^i}\right)\Psi\,,\end{aligned} \tag{C.20}$$

and

$$\frac{\partial\Psi}{\partial\overline{y}_*^{\bar{i}}} = \overline{C}_{\bar{i}\bar{j}\bar{k}}g(z_*)^{p\bar{j}}g(z_*)^{q\bar{k}}\frac{\partial^2\Psi}{\partial\eta_1^p\partial\eta_1^q} + \partial_j\bar{\partial}_{\bar{i}}\log k(y)\eta_1^j\left(y^0\frac{\partial}{\partial\eta_1^0} - \eta_1^0\frac{\partial}{\partial\eta_1^0} - \eta_1^i\frac{\partial}{\partial\eta_1^i}\right)\Psi\,.$$

These should be compared with the Holomorphic Anomaly equations [2].

In the rest of the appendix we briefly discuss the special Kähler geometry of the moduli space of the calibrated CY manifolds. Let M be a CY threefold and $H^3(M,\mathbb{C})$ be the complexified middle cohomology of M. Fix the symplectic basis $\{\gamma^i\}$ in $H_3(M,\mathbb{Z})$:

$$\left\langle\gamma_+^i,\gamma_j^-\right\rangle = \delta^{ij}\,,\qquad \left\langle\gamma_i^-,\gamma_j^-\right\rangle = 0\,,\qquad \left\langle\gamma_+^i,\gamma_+^j\right\rangle = 0\,.$$

The periods of a three-form ω

$$z^i = \int_{\gamma_+^i}\omega\,,\qquad w_i = \int_{\gamma_i^-}\omega\,, \tag{C.21}$$

provide the natural coordinates in $H^3(M,\mathbb{C})$.

There is a natural hyperkähler structure on $H^3(M,\mathbb{C})$ defined by the holomorphic two-form:

$$\omega^{2,0}(\delta_1\Omega,\delta_2\Omega) = \int_M\delta_1\Omega\wedge\delta_2\Omega\,, \tag{C.22}$$

and the Kähler form:

$$\omega^{1,1}(\delta_1\Omega,\delta_2\Omega) = \int_M\delta_1\Omega\wedge\delta_2\overline{\Omega}\,. \tag{C.23}$$

In terms of the coordinates (z^i,w_i) they are given by (C.7), (C.8).

Let $\widehat{\mathcal{M}}_M$ be the extended moduli space of the deformations of the calibrated CY manifold M. The period map defines the embedding $\widehat{\mathcal{M}}_M\to H^3(M,\mathbb{C})$ such that the image of the point corresponding to CY manifold M supplied with the holomorphic three-form Ω has coordinates $(\int_{\gamma_\pm^1}\Omega,\dots,\int_{\gamma_\pm^n}\Omega)$. It may be shown that the image of $\widehat{\mathcal{M}}_M$ is actually a lagrangian sub-manifold in $H^3(M,\mathbb{C})$ with respect to the holomorphic symplectic structure (C.22) and thus in the coordinates (C.21) may be described in terms of some holomorphic function $\mathcal{F}(z)$ as follows:

$$z^i = \int_{\gamma_i^+}\Omega\,,\qquad w_i = \int_{\gamma_i^-}\Omega = \frac{\partial\mathcal{F}(z)}{\partial z^i}\,.$$

Given a local coordinates x^a on the moduli space $\widehat{\mathcal{M}}$ the set of the three-forms Ω, $\nabla_a\Omega$, $\overline{\nabla_a\Omega}$, $\overline{\Omega}$ provide a basis in the complexified tangent space to $\widehat{\mathcal{M}}$ (as in section 1 we identify

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$\Omega^{-p,q}$ and $\Omega^{3-p,q}$ with the help of the holomorphic three-form Ω). Taking into account that the complexified tangent space to $\widehat{\mathcal{M}}_M$ may be identified with $H^3(M,\mathbb{C})$ we get a family of the bases in $H^3(M,\mathbb{C})$ parameterized by $\widehat{\mathcal{M}}_M$.

Now let us define the coordinates (η,ξ) on the tangent space to the point $(z_*,\bar{z}_*) \in \widehat{\mathcal{M}}_M$ as:

$$\begin{aligned} G_{a\overline{a}}\eta^a &= \int_M (\omega(z) - \Omega(z_*)) \wedge \overline{\nabla_a \Omega}\,, \\ \xi_a &= \int_M (\omega(z) - \Omega(z_*)) \wedge \nabla_a \Omega\,, \end{aligned}$$

where G is a Kähler metric associated with the Kähler form (C.23). It is easy to show that these coordinates are the special case of the coordinates (C.11), (C.12) introduced above and the KS action on the classical solutions takes the form:

$$S_{cl} = \int_M \Omega(x|z_*,\overline{z}_*) \wedge \Omega(0|z_*,\overline{z}_*) = \sum_i (z^i w_i(z_*) - w_i(z) z^i_*)\,. \tag{C.24}$$

It is useful to specialize this representation to the limiting case of the choice of $(z_*,\bar{z}_*)$ at the point of the maximal degeneration of the complex structure $(z_\infty,\bar{z}_\infty)$ (see e.g. [12]). In this case we have $\lim_{z_*\to z_\infty}\Omega = [\gamma_0]^\vee$ where $[\gamma_0]$ is the three-form dual to the invariant cycle γ_0 and the action takes the simple form:

$$S_{cl} = \int_M \Omega(x|z_*,\overline{z}_*) \wedge [\gamma_0]^\vee = \int_{\gamma_0} \Omega(x|z_*,\overline{z}_*)\,. \tag{C.25}$$

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